

Critical point from scaling of electron charge in 3D Dirac semimetals

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We investigate the effects of the scaling of the electron charge in the Quantum Electrodynamics of 3D Dirac semimetals, which arise as electron-hole excitations become less efficient to screen the bare charge at short-distance scales. We show that these 3D electron systems have indeed a critical point for values of the effective coupling corresponding to the instance in which the bare coupling gets very large at the high-energy cutoff. The critical behavior is characterized by the suppression of the quasiparticle weight at low energies, making the system to fall then into the class of marginal Fermi liquids. We deal also with the investigation of the phase beyond the critical point, carrying out a self-consistent resolution of the Schwinger-Dyson equations of the electron system. We find that this can be still characterized in that regime in terms of electron quasiparticles, but with parameters that have large imaginary parts, up to a limit interaction strength marking the breakdown of the homogeneous formulation of the theory over long-distance scales.

Introduction.— The discovery of graphene has opened new avenues of research in both theoretical and applied physics. The remarkable electronic properties of the material come to a great extent from the peculiar conical dispersion of the electron quasiparticles, which endows them with an additional pseudospin quantum number[1]. This is an example of so-called Dirac semimetal, which provides an ideal playground to test many of the properties typical of relativistic fermion fields, like the Klein paradox[2] or the anomalous screening of charged particles[3–6].

Another remarkable feature of relativistic field theories is the running of the coupling constants with the energy scale at which the system is measured. In this regard, Dirac semimetals like graphene are a kind of Quantum Electrodynamics (QED) in two spatial dimensions. While the electron charge remains invariant in such a theory, the effective interaction strength is not constant however, as a result of the dependence on energy of the Fermi velocity. This behavior has been actually measured in suspended graphene samples at very low doping levels[7], showing that the Fermi velocity grows as predicted[8, 9] when looking close to the Dirac point (the vertex of the conical dispersion).

The effective interaction strength flows monotonously in graphene towards a free fixed-point at low energies, which may explain the absence of significant electronic correlations in the carbon sheet. The recent discovery of materials with linear electronic dispersion in three dimensions[10–13] opens however the possibility of finding more exotic behaviors, stemming from the properties of QED in such a higher dimension. In that theory, the electron charge e is screened at long distances by electron-hole pairs so that its value runs with the energy scale μ , being related to the bare charge e_Λ at the high-energy cutoff Λ through the expression[14]

$$e_\Lambda^2 = \frac{e^2(\mu)}{1 - \frac{1}{6\pi^2 c} e^2(\mu) \log \frac{\Lambda}{\mu}} \quad (1)$$

For constant e_Λ , this implies that the measurable charge $e(\mu)$ flows to zero at low energies, leading asymptotically to the noninteracting theory.

Assuming conversely that e has some finite value at energy μ , the above equation shows that the bare coupling e_Λ should blow up at a certain value of the large cutoff Λ . This is the well-known Landau pole[14], that for some time cast many doubts about the quantum field theory approach to the description of elementary particles, given the impossibility to attach any physical meaning to such a high-energy singularity. In the QED of 3D Dirac semimetals, however, the high-energy cutoff Λ is a magnitude that can be related to the short-distance scale of the microscopic lattice, so it makes sense to ask about the influence of the Landau pole in those systems. In particular, there should be no fundamental obstruction to reach the regime of very large coupling constant in a material with suitably strong e - e interaction, whose effective strength is in general given by the ratio between e^2 and the Fermi velocity v_F of the electron quasiparticles.

In the present paper, we investigate the effects implied by the Landau pole in the QED of 3D Dirac semimetals, in which the speed of light c is replaced by a much smaller Fermi velocity v_F . Taking formally the limit of a large number N of fermion flavors, we will see that such a theory has a critical point in the effective interaction strength $g \equiv Ne^2/2\pi^2 v_F$ at $g_c = 3$. This critical coupling will be obtained in a rigorous scale-invariant calculation of the electron scaling dimension, showing the vanishing of the electron quasiparticle weight at the critical point. A similar result will be also obtained by the self-consistent resolution of the Schwinger-Dyson equation for the electron propagator. This approach demands the introduction of an explicit high-energy cutoff in the formulation of the theory, which will reveal that the onset of non-Fermi liquid behavior coincides with the point at which the effective interaction becomes large enough to drive the bare coupling into the Landau pole at the cutoff Λ .

In the condensed matter context, it is also admissible to ask about the physics beyond the critical point, envisaging the possibility of electron systems where the effective coupling is larger than the critical value g_c . We will see that in these conditions we are led into a strongly renormalized phase, where the quasiparticle parameters get large imaginary parts. This “dirty metal” phase is in turn the precursor of a limit interaction strength g^* beyond which the real part of the quasiparticle weight becomes negative at low momenta, signaling the breakdown of the homogeneous formulation of the 3D Dirac semimetal over long-distance scales.

Scaling properties of 3D Dirac semimetals.— We describe the QED of non-relativistic Dirac fermions starting from the hamiltonian for a collection of N four-component Dirac spinors $\{\psi_i\}$ in a generic spatial dimension D

$$H = iv_F \int d^D r \psi_i^\dagger(\mathbf{r}) \gamma_0 \gamma \cdot \nabla \psi_i(\mathbf{r}) + e_0 \int d^D r \psi_i^\dagger(\mathbf{r}) \psi_i(\mathbf{r}) \phi(\mathbf{r}) \quad (2)$$

where ϕ stands for the scalar potential and $\{\gamma_\alpha\}$ is a set of Dirac matrices satisfying $\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}$. Here η represents the Minkowski metric, guaranteeing that the kinetic term in the hamiltonian has eigenvalues $\pm v_F |\mathbf{k}|$ in momentum space. The physical dimension corresponds to $D = 3$, but we will start shifting formally the dimension to $D = 3 - \epsilon$ in order to regularize the divergences that the theory has in the limit of high energy and momentum, following a scheme aimed to preserve the gauge invariance in the computation of observable quantities[15, 16].

In our non-relativistic theory, ϕ mediates the Coulomb interaction between electrons and it has a free propagator in momentum space $D_0(\mathbf{q}, \omega) = e_0^2 / \mathbf{q}^2$. This is corrected by the electron-hole polarization $\Pi(\mathbf{q}, \omega)$, which is a divergent quantity at $D = 3$ [17]. Computing to leading order in a $1/N$ expansion, we get the expression for the ϕ propagator

$$D(\mathbf{q}, \omega) = \frac{e_0^2}{\mathbf{q}^2 + NB(\epsilon) \frac{e_0^2}{2\pi^2 v_F} \frac{\mathbf{q}^2}{(v_F^2 \mathbf{q}^2 - \omega^2)^{(\epsilon/2)}}} \quad (3)$$

with $B(\epsilon) = (4\pi)^{\epsilon/2} \Gamma(\epsilon/2) \Gamma(2 - \epsilon/2) / \Gamma(4 - \epsilon)$. The divergence as $\epsilon \rightarrow 0$ can be reabsorbed into a simple renormalization of the bare electron charge e_0 , passing to the physical dimensionless coupling e with the help of an auxiliary energy scale μ through the expression $\mu^\epsilon / e_0^2 = 1/e^2 - N/6\pi^2 v_F \epsilon$. We have then

$$e_0^2 = \frac{\mu^\epsilon e^2}{1 - \frac{N}{6\pi^2 v_F} e^2 \frac{1}{\epsilon}} \quad (4)$$

which is the counterpart of Eq. (1) in the dimensional regularization approach, where the $\log(\Lambda)$ dependence is replaced by the $1/\epsilon$ pole.

We end up in this way with an expression of the ϕ

propagator which is finite in the limit $\epsilon \rightarrow 0$,

$$D(\mathbf{q}, \omega) = \frac{\mu^\epsilon e^2}{\mathbf{q}^2 \left(1 - \frac{N e^2}{6\pi^2 v_F} \frac{1}{\epsilon} + NB(\epsilon) \frac{e^2}{2\pi^2 v_F} \frac{\mu^\epsilon}{(v_F^2 \mathbf{q}^2 - \omega^2)^{(\epsilon/2)}} \right)} \quad (5)$$

Right at $\epsilon = 0$, the denominator of (5) has a logarithmic dependence on $v_F^2 \mathbf{q}^2 - \omega^2$. Thus, the propagator blows up at sufficiently large values of that quantity, which is again a manifestation of the Landau pole. We will exploit the scale invariance of the theory (so-called renormalizability, that we check to leading order in the $1/N$ expansion) to assess the effect of the pole on physical quantities, that turn out to be pure functions of the renormalized coupling e , with no dependence on the auxiliary scale μ in the limit $\epsilon \rightarrow 0$.

We then study the effect of the Landau pole on electron quasiparticle properties. For that purpose, one can compute the electron self-energy $\Sigma(\mathbf{k}, \omega_k)$, which is given to leading order of the $1/N$ expansion by

$$i\Sigma(\mathbf{k}, \omega_k) = \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_p}{2\pi} G_0(\mathbf{k} - \mathbf{p}, \omega_k - \omega_p) D(\mathbf{p}, \omega_p) \quad (6)$$

where $G_0(\mathbf{p}, \omega_p)$ stands for the free Dirac propagator. The self-energy develops its own divergences in the limit $\epsilon \rightarrow 0$, which can be completely absorbed into a redefinition of quasiparticle parameters by renormalization factors Z_ψ and Z_v in the expression of the propagator

$$G(\mathbf{k}, \omega_k)^{-1} = Z_\psi(\omega_k - Z_v v_F \gamma_0 \gamma \cdot \mathbf{k}) - Z_\psi \Sigma(\mathbf{k}, \omega_k) \quad (7)$$

The renormalization factors have the pole structure $Z_\psi = 1 + (1/N) \sum_{n=1}^\infty c_n(g)/\epsilon^n$, $Z_v = 1 + (1/N) \sum_{n=1}^\infty b_n(g)/\epsilon^n$ in terms of the effective coupling $g \equiv N e^2 / 2\pi^2 v_F$. The important point is that, in the present theory, the electron propagator G can be made free of poles in the ϵ variable with an appropriate choice of coefficients $c_n(g)$ and $b_n(g)$ which do not depend on the auxiliary scale μ (see Appendix). This is a crucial property, since the electron correlators get in general anomalous scaling dimensions[18] that are given by powers of

$$\gamma_d = \frac{\mu}{Z_\psi} \frac{\partial Z_\psi}{\partial \mu} \quad (8)$$

Under the rescaling $\mathbf{k} \rightarrow s\mathbf{k}$, $\omega \rightarrow s\omega$, the electron propagator becomes for instance

$$G(s\mathbf{k}, s\omega) \approx s^{-1+\gamma_d} G(\mathbf{k}, \omega) \quad (9)$$

In a theory scaling perfectly as in the present instance, the only dependence of Z_ψ on μ comes from the dependence implicit in the coupling g , leading to $\gamma_d = -g'_1(g)/N$ [19], which allows to obtain such an observable quantity exclusively in terms of the physical coupling g .

We have computed the coefficient $c_1(g)$ up to very high orders in the coupling g , finding that these approach a

precise geometric growth. This means that the power series in g has a finite radius of convergence, which we have determined to be at $g_c = 3$ (see Appendix). The main consequence of this behavior of $c_1(g)$, displayed in Fig. 1(a), is the divergence of the anomalous exponent γ_d at such a critical coupling. According to (9), this has to be interpreted as the suppression of the quasiparticle weight in the low-energy limit.

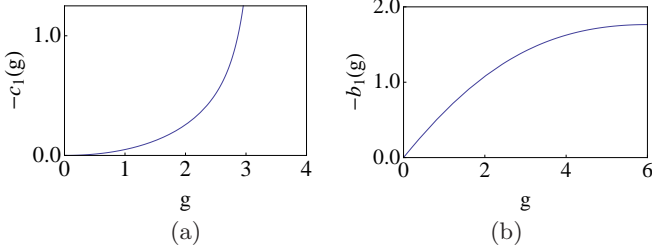


FIG. 1. Plot of the first residues $c_1(g)$ and $b_1(g)$ in the renormalization factors Z_ψ and Z_v , dictating respectively the scaling of the quasiparticle weight and the Fermi velocity at low energies.

One can check that the renormalized Fermi velocity has instead a regular behavior at the critical point. This can be seen from inspection of the residues of the poles in Z_v , that remain finite at g_c (see Appendix). The condition of independence of the bare Fermi velocity on the auxiliary scale, $\mu \partial(Z_v v_F)/\partial \mu = 0$, leads to the scaling equation for the renormalized Fermi velocity

$$\mu \frac{\partial v_F}{\partial \mu} = \frac{1}{N} g b'_1(g) v_F \quad (10)$$

$b'_1(g)$ is a negative bounded function up to g_c , as seen in Fig. 1(b), giving rise therefore to a limited growth of the Fermi velocity in the low-energy limit $\mu \rightarrow 0$. We conclude then that the singularity found at the critical coupling does not produce a qualitative change in the electronic dispersion, but rather translates into a strong attenuation of the own electron quasiparticles.

While the critical point is given by the value g_c of the effective coupling, we have to bear in mind that e^2 has a logarithmic dependence of the type implied by Eq. (1). This means that whether the critical behavior sets in or not in a particular system will be dictated by the maximum value reached by the effective coupling g . That will depend on characteristics like the particular Fermi velocity and number of Dirac cones of the material under consideration. The physics turns out to be quite different in the two situations since hitting the singularity at g_c implies the destabilization of the Fermi liquid regime, as we will see below with more detail.

Beyond the critical point.— Although we have been able to identify and characterize the critical point at g_c , the previous approach does not allow us to ask about the regime beyond the critical coupling. In order to access

that phase of the system, we adopt an alternative approach, that consists in the self-consistent resolution of the Schwinger-Dyson equation for the electron propagator. To make the parallel with the results in the previous section, we will neglect vertex corrections and characterize the quasiparticle properties in terms of the functions $z_\psi(\mathbf{k}, \omega)$, $z_v(\mathbf{k}, \omega)$ and $z_m(\mathbf{k}, \omega)$, writing the electron propagator in the form

$$G(\mathbf{k}, \omega) = (z_\psi(\mathbf{k}, \omega)\omega - z_v(\mathbf{k}, \omega)v_F\gamma_0\boldsymbol{\gamma} \cdot \mathbf{k} - z_m(\mathbf{k}, \omega)\gamma_0)^{-1} \quad (11)$$

The function $z_m(\mathbf{k}, \omega)$ is now introduced to study the possible dynamical generation of a mass term (and the consequent gap) at the Dirac point, assuming that the original theory does not have such a bare coupling in the hamiltonian.

The resolution proceeds then by computing the interaction propagator $D(\mathbf{q}, \omega)$ as in Eq. (3) with the polarization

$$i\Pi(\mathbf{q}, \omega_q) = \int \frac{d^3p}{(2\pi)^3} \frac{d\omega_p}{2\pi} G(\mathbf{q}-\mathbf{p}, \omega_q-\omega_p) G(\mathbf{p}, \omega_p) \quad (12)$$

while adjusting the functions $z_\psi(\mathbf{k}, \omega)$, $z_v(\mathbf{k}, \omega)$ and $z_m(\mathbf{k}, \omega)$ to attain self-consistency in the evaluation of the electron propagator $G(\mathbf{k}, \omega_k)$ with the self-energy

$$i\Sigma(\mathbf{k}, \omega_k) = \int \frac{d^3p}{(2\pi)^3} \frac{d\omega_p}{2\pi} G(\mathbf{k}-\mathbf{p}, \omega_k-\omega_p) D(\mathbf{p}, \omega_p) \quad (13)$$

The main difference with respect to the previous regularization is that now the computation of the integrals in (12) and (13) requires the introduction of a high-energy cutoff Λ . Accordingly, in order to account for the scaling of the charge with the energy as measured from the cutoff, we introduce the physical coupling $e(\mu)$ as in Eq. (1). In practice, a rotation has to be performed for all the frequencies in the complex plane, $\omega = i\bar{\omega}$, passing then to a Euclidean space for the variables $(\mathbf{k}, \bar{\omega})$. In this approach, we can access different regimes of the theory depending on whether the coupling $g = Ne^2/2\pi^2 v_F$ is large enough to reach the pole in $D(\mathbf{p}, i\bar{\omega}_p)$ starting from the bare electron propagator.

The regular Fermi liquid regime of the theory corresponds to the case where the pole is not hit at any frequency and momentum up to the cutoff Λ . Then, a purely real solution exists for $z_\psi(\mathbf{k}, i\bar{\omega})$ and $z_v(\mathbf{k}, i\bar{\omega})$, while $z_m(\mathbf{k}, i\bar{\omega})$ turns out to be always self-consistently set to zero. In this approach one also finds however a critical point g_c , characterized by the divergence of $z_\psi(\mathbf{k}, i\bar{\omega})$ in the limit of vanishing frequency, while the renormalization factor $z_v(\mathbf{k}, i\bar{\omega})$ still has a smooth behavior, as shown in Fig. 2. The critical behavior is then governed by the vanishing of the quasiparticle weight at low energies, in clear correspondence with the singular wave-function renormalization found in the dimensional regularization of the theory.

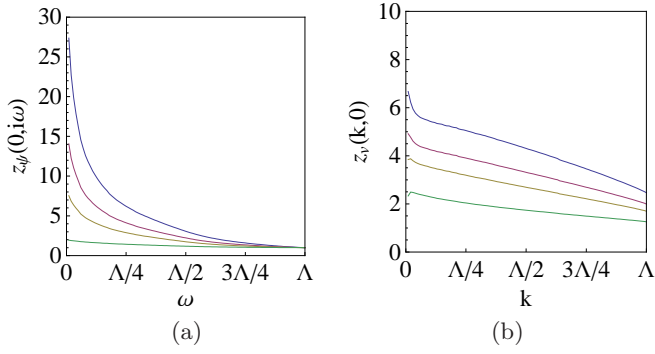


FIG. 2. Plot of the renormalization factors $z_\psi(0, i\omega)$ and $z_v(\mathbf{k}, 0)$ for $N = 1$ and renormalized coupling (curves from top to bottom) $e^2(\mu)/2\pi^2 v_F = 4.328, 4.3, 4.2, 3.0$, referred to the energy scale $\mu = \Lambda/2$.

We note however that, as a difference with respect to the previous scaling approach, the self-consistent resolution leads now to the identification of the critical behavior at values of the electron charge $e(\mu)$ referred to a concrete energy scale μ . At $\mu = \Lambda/2$, for instance, the critical coupling turns out to be given (for $N = 1$) by $e^2(\mu)/2\pi^2 v_F \approx 4.328$. Remarkably, this corresponds to the orbit producing the divergence of the bare coupling at the endpoint $\mu = \Lambda$ according to the rule (1) (with c replaced by v_F).

The interesting point about the present approach is that it allows to investigate the properties of the theory beyond the critical point g_c . The self-consistent resolution of the Schwinger-Dyson equation can be carried out for $g > g_c$ in the same fashion as before, with the result that $z_\psi(\mathbf{k}, i\bar{\omega})$ and $z_v(\mathbf{k}, i\bar{\omega})$ become now complex functions. We can ascribe this new behavior to the onset of a different phase of the electron system, in which electron quasiparticles still exist but with a decay rate dictated by the imaginary contributions in the self-energy. These may get very large and comparable to the own real part in $\Sigma(\mathbf{k}, \omega_k)$ at low energies, as shown in Fig. 3, meaning that we are dealing in this regime with a kind of “dirty” metal with very unstable quasiparticles near the Dirac point.

A last important feature regarding the self-consistent resolution is that, beyond a certain coupling, it can be observed a change in the monotonous decreasing trend of the quasiparticle weight at low frequencies. This behavior is actually the precursor leading to a new critical coupling g^* at which the function $z_\psi(\mathbf{k}, \bar{\omega})$ vanishes at $\bar{\omega} = 0$ and $\mathbf{k} = 0$, as seen in Fig. 3(a), to become negative for $g > g^*$ in the neighborhood of that point. This has to be understood as the signature of yet a different phase, whose description exceeds however the framework of our many-body formulation, as one cannot make physical sense of negative values in the real part of the quasiparticle parameters. We interpret therefore that the

electron system cannot be cast in terms of a set of continuous fields beyond the coupling g^* , which is supported by the fact that the anomaly (the negative values in the quasiparticle weight) is confined to a range about $\omega = 0$ and $\mathbf{k} = 0$ that grows at larger couplings. Concomitant with that feature is the fact that the dressed electron-hole polarization starts to develop a pole at the high-energy part of the spectrum near Λ , pointing also at the phase separation of the system into different domains in the real space.

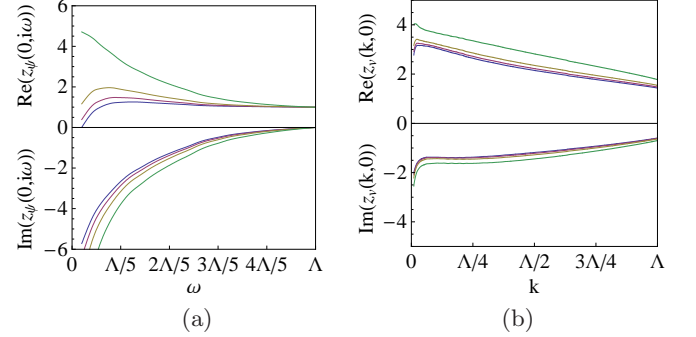


FIG. 3. Plot of the real and imaginary parts of the renormalization factors $z_\psi(0, i\omega)$ and $z_v(\mathbf{k}, 0)$ for $N = 1$ and renormalized coupling (curves from lower to higher absolute value in both sides) $e^2(\mu)/2\pi^2 v_F = 4.668, 4.6, 4.52, 4.4$, referred to the energy scale $\mu = \Lambda/2$.

Conclusion.— We have seen that, unlike 2D Dirac semimetals that have a dominant instability towards exciton condensation at strong coupling, their 3D analogues have instead a critical point characterized by the suppression of the electron quasiparticle weight at low energies. This happens without the development of a gap in the electronic spectrum, thus providing a genuine example within the class of the so-called marginal Fermi liquids, introduced many years ago in the effort to understand the anomalies of the normal state of copper-oxide superconductors.

Our analysis has been based on the scaling properties of relativistic field theories, in which the values of the coupling constants depend in general on the energy at which they are measured. We have found that the above mentioned marginal Fermi liquid behavior arises from the scaling of the electron charge, which grows large as electron-hole excitations become less efficient to screen the bare charge at short distances.

We have also investigated the phase arising when the effective coupling is larger than the critical coupling g_c . Our self-consistent resolution of the Schwinger-Dyson equations has led us to predict a phase with growing compressibility and the breakdown of the homogeneous formulation of the many-body theory, providing definite signatures susceptible of being confirmed by the experimental observation of the 3D Dirac semimetals in such a

strong coupling regime.

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Appendix

The electron self-energy computed to leading order in the $1/N$ expansion is given by the expression

$$i\Sigma(\mathbf{k}, \omega_k) = \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_p}{2\pi} \frac{\omega_k - \omega_p + v_F \gamma_0 \boldsymbol{\gamma} \cdot (\mathbf{k} - \mathbf{p})}{v_F^2 (\mathbf{k} - \mathbf{p})^2 - (\omega_k - \omega_p)^2 - i\eta} \frac{\mu^\epsilon e^2}{\mathbf{p}^2 \left(1 - \frac{Ne^2}{6\pi^2 v_F} \frac{1}{\epsilon} + NB(\epsilon) \frac{e^2}{2\pi^2 v_F} \frac{\mu^\epsilon}{(v_F^2 \mathbf{p}^2 - \omega_p^2)^{\epsilon/2}} \right)} \quad (14)$$

with $B(\epsilon) = (4\pi)^{\epsilon/2} \Gamma(\epsilon/2) \Gamma(2 - \epsilon/2)^2 / \Gamma(4 - \epsilon)$. In order to obtain the real part of $\Sigma(\mathbf{k}, \omega_k)$, we can perform a rotation to imaginary frequencies $\omega = i\bar{\omega}$. The self-energy becomes then, as a function of the effective coupling $g = Ne^2/2\pi^2 v_F$,

$$\Sigma(\mathbf{k}, i\bar{\omega}_k) = \frac{2\pi^2}{N} v_F^\epsilon \int \frac{d^D p}{(2\pi)^D} \frac{d\bar{\omega}_p}{2\pi} \frac{i\bar{\omega}_k - i\bar{\omega}_p + \gamma_0 \boldsymbol{\gamma} \cdot (v_F \mathbf{k} - \mathbf{p})}{(v_F \mathbf{k} - \mathbf{p})^2 + (\bar{\omega}_k - \bar{\omega}_p)^2} \frac{g\mu^\epsilon}{\mathbf{p}^2 \left(1 - g\frac{1}{3\epsilon} + gB(\epsilon) \frac{\mu^\epsilon}{(\mathbf{p}^2 + \bar{\omega}_p^2)^{\epsilon/2}} \right)} \quad (15)$$

In the analytic continuation to dimension $D = 3 - \epsilon$, the divergences that the self-energy develops at high energies appear as powers of $1/\epsilon$. One has to check that these poles can be absorbed into appropriate renormalization factors Z_ψ and Z_v , rendering convergent at $D = 3$ the electron propagator $G(\mathbf{k}, \omega_k)$ given by

$$G^{-1} = Z_\psi(\omega_k - Z_v v_F \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{k}) - Z_\psi \Sigma \quad (16)$$

The renormalization factors must have the pole structure

$$Z_\psi = 1 + \frac{1}{N} \sum_{n=1}^{\infty} \frac{c_n(g)}{\epsilon^n} \quad (17)$$

$$Z_v = 1 + \frac{1}{N} \sum_{n=1}^{\infty} \frac{b_n(g)}{\epsilon^n} \quad (18)$$

The residues $c_n(g)$ and $b_n(g)$ can be obtained in the form of power series in the g coupling constant starting from the expansion

$$\Sigma(\mathbf{k}, i\bar{\omega}_k) = \frac{2\pi^2}{N} v_F^\epsilon \int \frac{d^D p}{(2\pi)^D} \frac{d\bar{\omega}_p}{2\pi} \frac{i\bar{\omega}_k - i\bar{\omega}_p + \gamma_0 \boldsymbol{\gamma} \cdot (v_F \mathbf{k} - \mathbf{p})}{(v_F \mathbf{k} - \mathbf{p})^2 + (\bar{\omega}_k - \bar{\omega}_p)^2} \frac{g\mu^\epsilon}{\mathbf{p}^2} \sum_{n=0}^{\infty} (-1)^n g^n \left(-\frac{1}{3\epsilon} + B(\epsilon) \frac{\mu^\epsilon}{(\mathbf{p}^2 + \bar{\omega}_p^2)^{\epsilon/2}} \right)^n \quad (19)$$

To compute for instance the renormalization of the quasiparticle weight, we can set $\mathbf{k} = 0$ and use systematically the result

$$\int \frac{d^D p}{(2\pi)^D} \frac{d\bar{\omega}_p}{2\pi} \frac{i\bar{\omega}_k - i\bar{\omega}_p}{\mathbf{p}^2 + (\bar{\omega}_k - \bar{\omega}_p)^2} \frac{1}{\mathbf{p}^2} \frac{1}{(\mathbf{p}^2 + \bar{\omega}_p^2)^{m\epsilon/2}} =$$

$$i\bar{\omega}_k \frac{1}{(4\pi)^{2-\epsilon/2}} \frac{m\epsilon}{1-\epsilon} \frac{\Gamma(\frac{m+1}{2}\epsilon)\Gamma(1-\frac{m+1}{2}\epsilon)\Gamma(1-\frac{\epsilon}{2})}{\Gamma(1+\frac{m}{2}\epsilon)\Gamma(2-\frac{m+2}{2}\epsilon)} \frac{1}{|\bar{\omega}_k|^{(m+1)\epsilon}} \quad (20)$$

In this way we obtain the analytic expression of the first orders of the residues $c_n(g)$

$$c_1(g) = -\frac{1}{24}g^2 - \frac{1}{192}g^3 - \frac{5}{5184}g^4 - \left(\frac{1}{6480} + \frac{\zeta(3)}{6480}\right)g^5 - \left(\frac{7}{279936} + \frac{\pi^4}{2799360} + \frac{\zeta(3)}{34992}\right)g^6$$

$$- \left(\frac{1}{244944} + \frac{\pi^4}{14696640} + \frac{5\zeta(3)}{979776} + \frac{\zeta(5)}{108864}\right)g^7 + \dots \quad (21)$$

$$c_2(g) = -\frac{1}{108}g^3 - \frac{1}{648}g^4 - \frac{1}{3888}g^5 - \left(\frac{1}{23328} + \frac{\zeta(3)}{23328}\right)g^6 - \left(\frac{1}{139968} + \frac{\pi^4}{9797760} + \frac{\zeta(3)}{122472}\right)g^7 + \dots \quad (22)$$

$$c_3(g) = -\frac{1}{432}g^4 - \frac{1}{2430}g^5 - \frac{5}{69984}g^6 - \left(\frac{1}{81648} + \frac{\zeta(3)}{81648}\right)g^7 + \dots \quad (23)$$

$$c_4(g) = -\frac{1}{1620}g^5 - \frac{1}{8748}g^6 - \frac{5}{244944}g^7 + \dots \quad (24)$$

$$c_5(g) = -\frac{1}{5832}g^6 - \frac{1}{30618}g^7 + \dots \quad (25)$$

$$c_6(g) = -\frac{1}{20412}g^7 + \dots \quad (26)$$

A most important feature regarding these expressions is that they do not contain any logarithmic dependence (in fact any dependence) on $\bar{\omega}_k$, which is crucial to guarantee the interpretation of Z_ψ as the renormalization of a local operator in the original action of the theory.

The reiterated use of (20) affords anyhow a deeper numerical investigation, by which $c_n(g)$ can be computed to much higher order in perturbation theory. Thus, we have been able to obtain the coefficients of the expansion

$$c_1(g) = \sum_{n=1}^{\infty} c_1^{(n)} g^n \quad (27)$$

up to order g^{32} . As evidenced in the plot of Fig. 4, the expansion (27) approaches a geometric series at large n , which means that it must have a finite radius of convergence in the variable g . An excellent fit of the n -dependence of $c_1^{(n+1)}/c_1^{(n)}$ is achieved by assuming the scaling behavior

$$\frac{c_1^{(n)}}{c_1^{(n-1)}} = r + \frac{r'}{n} + \frac{r''}{n^2} + \frac{r'''}{n^3} + \dots \quad (28)$$

We get in this way an accurate estimate

$$r \approx 0.3333333 \quad (29)$$

leading to the radius of convergence $g_c = 1/r \approx 3.0 \pm 1.0 \times 10^{-7}$.

A similar approach can be used to obtain the expansion of the residues $b_n(g)$ in the renormalization factor Z_v for the Fermi velocity. In this case, the analytic computation of the first perturbative orders leads to the expressions

$$b_1(g) = -c_1(g) - \frac{1}{3}g - \frac{1}{72}g^2 - \frac{1}{324}g^3 - \left(\frac{1}{1728} + \frac{\zeta(3)}{1296}\right)g^4 - \left(\frac{1}{9720} + \frac{\pi^4}{583200} + \frac{\zeta(3)}{19440}\right)g^5 + \dots \quad (30)$$

$$b_2(g) = -c_2(g) - \frac{1}{18}g^2 - \frac{1}{324}g^3 - \frac{1}{1296}g^4 - \left(\frac{1}{6480} + \frac{\zeta(3)}{4860}\right)g^5 + \dots \quad (31)$$

$$b_3(g) = -c_3(g) - \frac{1}{81}g^3 - \frac{1}{1296}g^4 - \frac{1}{4860}g^5 + \dots \quad (32)$$

$$b_4(g) = -c_4(g) - \frac{1}{324}g^4 - \frac{1}{4860}g^5 + \dots \quad (33)$$

$$b_5(g) = -c_5(g) - \frac{1}{1215}g^5 + \dots \quad (34)$$

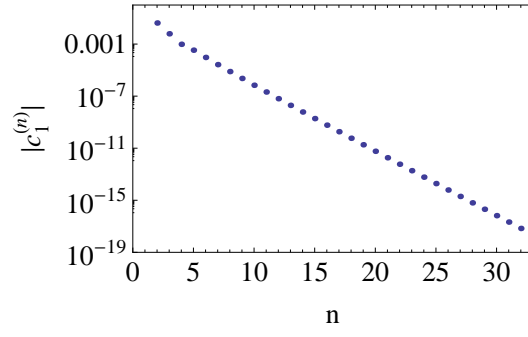


FIG. 4. Plot of the absolute value of the coefficients $c_1^{(n)}$ in the expansion of $c_1(g)$ as a power series of the coupling g .

It can be seen that the coefficients in these expansions do not have now a constant sign within each of the power series. One can check that these define indeed smooth functions in the range up to the coupling g_c obtained above, implying that the scaling of the Fermi velocity has a regular behavior in the Dirac semimetal.